Some equations of special relativity

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Einstein's theory of special relativity is based on two assumptions:

1. All inertial (i.e. non-accelerating) frames of reference are equally valid (i.e. any observations or experiments performed will produce equally valid results). 2. The speed of light is constant for all inertial frames of reference.

Imagine a spacecraft passing the Earth with velocity v. On the spacecraft is an observer and an apparatus that will flash a beam of light across the spacecraft (perpendicular to the spacecraft's motion). On the Earth is another observer.

According to the observer on the spacecraft, the light beam travels a distance w, where w is the width of the spacecraft. However, the observer of Earth will see the light beam cover a greater distance, due to the motion of the spacecraft while the light beam is en route. If t is the time the light beam takes to cross the spacecraft, then the spacecraft travels distance vt in this time. Then the distance travelled by the light beam is

$$d = \sqrt{w^2 + v^2 t^2} \tag{1}$$

Now in pre-relativistic (Newtonian) physics, both observers record the same period of time. Thus, the velocity recorded by the two observers is different: the Earth-bound observer would record a greater velocity for the beam of light.

However, Einstein's assumption is that the velocity of light is the same for both observers.

The speed of light, c, would be calculated by the observer on the spacecraft as:

$$c = \frac{d'}{t'} = \frac{w}{t'} \tag{2}$$

Here, d' and t' are distance and time, respectively, measured on the spacecraft.

Solving for t' gives:

$$t' = \frac{w}{c} \tag{3}$$

The speed of light calculated by the observer on Earth is:

$$c = \frac{d}{t} = \frac{\sqrt{w^2 + v^2 t^2}}{t} = \sqrt{\frac{w^2}{t^2} + v^2} \tag{4}$$

Solving for t gives:

$$
\begin{aligned}
c^2 &= w^2 t^2 + v^2 \\
c^2 - v^2 &= w^2 t^2 \\
t^2(c^2 - v^2) &= w^2 \\
t^2 &= \frac{w^2}{c^2 - v^2} \\
t &= \frac{w}{\sqrt{c^2 - v^2}}
\end{aligned}
\tag{5}
$$

To find the ratio of time measured on the spacecraft to time measured on Earth, find t'/t:

$$
\begin{aligned}
\frac{t'}{t} &= \frac{w}{c} \div \frac{w}{\sqrt{c^2 - v^2}} \\
&= \frac{w}{c} \frac{\sqrt{c^2 - v^2}}{w} \\
&= \frac{\sqrt{c^2 - v^2}}{c} \\
&= \sqrt{\frac{c^2 - v^2}{c^2}} \\
\frac{t'}{t} &= \sqrt{1 - \left(\frac{v}{c}\right)^2}
\end{aligned}
\tag{6}
$$

If $v << c$, then the result t'/t is very close to 1. But as v approaches c, t'/t becomes smaller. Time slows down for the observer on the spacecraft as its speed approaches the speed of light!

The speed of light in vaccum (c) is 299,792.458 kilometers/second. Because c is so much larger than the speeds encountered in human experience (thus far!) the resultant time dilation is negligible. The fastest speed ever attained by humans (relative to the Earth) was 11.1 kilometers/second, by Apollo 10 astronauts returning from the Moon in 1969. For them, $t'/t = 0.9999999993$. This is equivalent to losing 1 second in 45 years.

At speeds on the order of the speed of light, this time dilation is significant. If $v/c = 0.6$, then $t'/t = 0.8$. If $v/c = 0.99$, then $t'/t = 0.141$.

Comparable transformations apply to the length of an object along the direction of travel and the object's mass:

$$
\frac{L'}{L} = \sqrt{1 - \left(\frac{v}{c}\right)^2} \tag{7}
$$

$$
\frac{m'}{m} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{8}
$$

Here, L and m are the length and mass, respectively, of an object at rest; L' and m' are the values for the moving reference frame.

Note that the length decreases and the mass of an object increases as the velocity approaches the speed of light. The increase in mass, however, is best understood not as a change in mass but as a change in the relationship of mass and momentum. In pre-relativistic physics, momentum p is described as the product of mass and velocity: $p = mv$. The relationship of mass and momentum in special relativity, however, is:

$$
p' = \frac{mv}{\sqrt{1 - (\frac{v}{c})^2}} \tag{9}
$$

Here, p' is the relativistic momentum and m is the rest mass.

This has some interesting consequences. First, for an object with non-zero mass as its velocity approaches the speed of light its momentum will approach infinity. This implies that that it would take an infinite amount of energy to accelerate an object with mass to the speed of light. Since this amount of energy is unavailable, objects with mass cannot travel at the speed of light (in vaccum) or faster. General relativity will expand on this understanding of the speed of light as a limiting speed in the universe.

Now we can show something else regarding the mass change. If a or b are (positive) numbers much less than one, then these approximations apply:

$$\begin{aligned} \frac{1}{1-a} &\simeq 1+a \\ \sqrt{1-b} &\simeq 1-\frac{1}{2}b \end{aligned} \tag{10}$$

If the speed is much smaller then the speed of light, or $v << c$, then we can apply these approximations to the above formula for m'/m and get:

$$m' \simeq m\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \tag{11}$$

If we multiply both sides by c^2, the result is:

$$m'c^2 \simeq mc^2 + \frac{1}{2}mv^2 \tag{12}$$

Notice that the second term on the right is the kinetic energy in pre-relativistic physics. The equation thus states that some amount of energy (mc^2) plus kinetic energy gives what we can call the relativistic energy of an object. Clearly, an object that is not moving has an associated amount of energy

$$E = mc^2 \tag{13}$$

This is Einstein's famous equation relating mass and energy. Special relativity concludes that mass and energy are equivalent concepts. (This is a better way to understand the mass/momentum issue discussed above, i.e. in terms of the increase in mass-energy of an object approaching the speed of light.)

`http://www.johnstonsarchive.net/relativity/specialrel.pdf`