Calculations on space-time curvature within the Earth and Sun

Wm. Robert Johnston

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1 Objective and disclaimer

One aspect of the relativistic space-time curvature due to the presence of mass is the breakdown of Euclidean (fiat-space) geometry. For example, the diameter of a spherical mass is greater than the result obtained by dividing the circumference by π. The objective of this article is to determine the amount of discrepancy between "relativistic" and" non-relativistic" diameters for objects such as the Sun and Earth.

This is the outcome of a small exercise I engaged in with my tiny understanding of relativity; serious relativists please forgive me (and set me straight as necessary!) (In this article equations in geometrized units arc denoted by an asterisk.)

2 Einstein's equation of general relativity (so you can say you've seen it)

The Einstein field equation can be written as

$$G_{\alpha\beta} = 8\pi \frac{G}{c^4} T_{\alpha\beta} \tag{1}$$

where $G_{\alpha\beta}$ is the Einstein curvature tensor, $T_{\alpha\beta}$ is the stress-energy tensor, G is the gravitational constant, and c is the spced of light. The Einstein tcnsor describes the curvature of space-time; the stress-energy tensor describes the density of mass-energy. This equation therefore concisely describes the curvature of space-time that results from the presence of mass-energy. This curvature in turn determines the motion of freely falling objects.

The math involved in using this equation to its fullest is a textbook-length subject. This equation can only be explicitly solved for limited situations, one of which is described by Schwarzschild.

3 The Schwarzschild metric and uniforrn density spheres

By extending the Pythogarean theorem one can write, for Euclidean (flat) space,

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{2}$$

In this relationship, ds is the distance between two nearby points separated by orthogonal displacements dx, dy, and dz.

Special relativity gives the metric

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{3}$$

which intertwines space and time. In other words, the separation ds of two nearby points is determined by separation in space and separation in time.

The Schwarzschild metric expresses this as

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad * \tag{4}$$

Note that geometrized units are used here (and in all subsequent equations denoted by *), in which $c = G = 1$ such that $2M = R_S$, where R_S is the Schwarzschild radius. (Such practice is traditional in general relativity, which does some to indicates the degree to which theoretical work dominates in the field.) The Schwarzschild metric allows some explicit solutions for iwlated spherically symmetric objects. Specifically, for such an object the above metric may be written as

$$ds^2 = -e^{2\Phi}dt^2 + \frac{dr^2}{1 - 2m/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad * . \tag{5}$$

In this equation, m is is the mass within distance r of the center and can be written as

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr \tag{6}$$

where $\rho(r)$ is density as a function of r.

If we examine the space-time curvature on a radius of an object, then t, θ, and ϕ are constant. Then the Schwarzschild metric reduces to

$$ds^2 = \frac{dr^2}{1 - 2m/r} \qquad * \tag{7}$$

From this we get an expression for s, which is proper displacement (distance in the curved space-time) from the object's center to its surface:

$$s = \int_0^R \frac{dr}{\sqrt{1 - 2m/r}} \qquad * . \tag{8}$$

R is the Schwarzschild r-coordinate of the object's surface (taking the center to be the origin). More simply, R is the result of dividing the object's circumference by 2π. In contrast, s is the true distance from center to surface, or true radius.

If density ρ is assumed constant, then $m(r)$ can be written as

$$m(r) = \frac{r^3}{R^3} M \tag{9}$$

where R and M are the radius and total mass, respectively, of the object.

Combining these gives

$$s = \int_0^R \frac{dr}{\sqrt{1 - 2Mr^2/R^3}} \qquad * \,. \tag{10}$$

Note that the denominator in the expression is real. Since $2M < R$ for objects which are not black holes and $r^2 < R^2$ if we are constrained to the interior of the object. This assures that $(1 - (2M/R)(r^2/R^2))$ is positive.

Taking the integral gives

$$s = \sqrt{\frac{R^3}{2M}} \left[\arcsin\left(\frac{r}{\sqrt{R^3/2M}} \right) \right]_0^R \qquad * \,. \tag{11}$$

which evaluated gives

$$s = R\sqrt{\frac{R}{2M}} \arcsin\left(\sqrt{\frac{2M}{R}} \right) \qquad * \,. \tag{12}$$

If $2M$ is replaced by R_S and the equation is rewritten in conventional units, we obtain

$$s = \sqrt{\frac{3c^2}{8\pi G\rho}} \arcsin\left(\sqrt{\frac{R_S}{R}} \right). \tag{13}$$

This equation expresses s as dependent on the object's density and the degree to which is is close to being a black hole (the latter represented by R_S/R).

If the object's radius is much greater than its Schwarzschild radius R_S, or $R \gg 2M = R_S$, then the following approximation can be applied:

$$\arcsin x \simeq x + \frac{x^3}{6} \tag{14}$$

Applying this approximation to our previous result gives

$$s \simeq R + \frac{M}{3} = R + \frac{R_S}{6} = R + \Delta R \qquad * \,. \tag{15}$$

Note that this can be loosely interpreted to say that the "relativistic" radius s is equal to the Euclidean or "non-relativistic" radius R plus a correction ΔR. As long as $R \gg 2M$, the correction is dependent only on the object's mass and is independent of its radius.

4 Results for a uniform density Sun and Earth

To determine values of ΔR for the Sun and Earth, we use the following:

speed of light $c = 299{,}792{,}458$ m/s

gravitational constant $G = 6.6726 \times 10^{-11}$ m^3/kg s^2

geometrized mass $m = (G/c^2)m_{conv}$ (where m_{conv} = conventional mass)

The Schwarzschild radius which is $R_S = 2M$ in geometrized units is equal to the following in conventional units:

$$R_S = \frac{2GM}{c^2}. \tag{16}$$

The true circumference of a non-rotating black hole with a given mass is $C_{bh} = 2\pi R_S$

Specific observed and derived data for the Sun and Earth are as follows:

	Sun	Earth
GM	$1.32712438 \times 10^{20}$ m^3/s^2	$3.98600441 \times 10^{14}$ m^3/s^2
R	695,990 km	6,371.0 km
R_S	2.95325003 km	8.87005606 mm
ΔR	492 m	1.48 mm

The table lists GM rather than M, since GM for the Earth and Sun is known with greater accuracy than the mass. The measured radii of the Sun and Earth would correspond to R in our formulation, not s. For both bodies $R_S \ll R$, justifying use of the final expression of the relativistic correction to the radius as $\Delta R = M/3$–assuming uniform density.

5 The clash between model and reality: results for realistic models of the Sun and Earth

The Sun and Earth are not uniform in density. The density of both increases towards the center. Thc density at the center of the Sun is over 100 times its average density. The situation for the Earth is less extrcme where the inner core is 3.6 times as dense as the upper mantle.

The resultant concentration of mass toward the center will increase the space-time curvature. The results obtained above for uniform density models must therefore be called into question-particularly that for the Sun.

Obtaining a more realistic value for relativistic correction to radius requires a numerical integration of

$$s = \int_0^R \frac{dr}{\sqrt{1 - 2m(r)/r}} \qquad * \,. \tag{17}$$

while using a model mass distribution $m(r)/r$.

For the Sun the following interior model from Cox (ed., *Allen's Astrphysical Quantities*, 2000) was used:

r/R	$m(r)/M$
0	0
0.007	0.00003
0.02	0.001
0.09	0.057
0.22	0.399
0.32	0.656
0.42	0.817
0.52	0.908
0.60	0.945
0.71	0.977
0.81	0.992
0.91	0.999
0.96	0.9999
0.99	1.0000
1.00	1.0000

By extrapolating the model distribution shown in Figure I was obtained. This was used to numerically integrate (in an Excel spreadsheet) to obtain s using a step $dr = 8{,}000$ km. The result was $\Delta R = 2.05$ km, four timcs greater than for the uniform density model.

In the case of the Earth, an approximate relation $m(r)$ was derived from tabulated $\rho(r)$ values given in Lide (ed., *CRC Handbook of Chemistry and Physics*, 1997). The result for $m(r)$ is shown in Figure 2. In performing the numerical integration to obtain s, a variable step was used in conformance to the tabulated r and ρ values. As a result dr varied from 3 km to 300 km.

The result is a value of $\Delta R = 2.2$ mm, or 1.5 times the result for the uniform density model.

6 Relativistic volume for the Sun

Because of the curvature of space-time, the volume contained within the surface of the Sun and the Earth is greater than the volume enclosed by a similar surface in Euclidean space.

The relativistic volume V_{rel} is equal to

$$V_{rel} = \int_0^R 4\pi r^2 ds. \tag{18}$$

Using the same method as before, this was numerically integrated for the Sun. Volume was not directly compared to the result from $V = (4/3)\pi R^3$ because of the low accuracy of the integration method. This was instead compared to the result obtained when dr was substituted for ds (an expression for volume in Euclidean space). The accuracy is limited by the large step sii:e (for this reason a calculation for the Earth was not attempted).

For the Sun it was found that the ratio of V_{rel} to Euclidean volume was 1.000006. The relativistic correction to the Sun's volume is then about 6 ppm, or about 6 times the volume of the Earth.

7 Embedding diagram for the solar interior

Another expression of the space-time curvature is provided by the familiar embedding diagram. For this, $z(r)$ is the displacement ("lift-out") along a radius of the star, viewed as embedded in Euclidean (r, z) space.

For a uniform density object, Misner, Thorne, and Wheeler (*Gravitation*, 1973, p. 610) derive

$$z(r) = \sqrt{\frac{R^3}{2M}}\left(1 - \sqrt{1 - \frac{2Mr^2}{R^3}}\right) \qquad * \tag{19}$$

for $r \leq R$.

Using this expression s was obtained by numerical integration using the fact that $ds^2 = dr^2 + dz^2$ For the uniform density model, the result is $\Delta R = 0.5$ km, confirming the result from section 3. For the realistic solar interior model, z must be obtained by numerical integration, using the previous integration for s and the relation $dz = \sqrt{ds^2 - dr^2}$.

These results 'were used to compare $z(r)$ for the uniform density and realistic solar interior models. This is shown in Figure 3, where $z(R) \equiv 0$ for convenience. The upper curve is for the uniform density model and the lower curve for the realistic model. Note the different behavior of the curvature for the realistic model. The difference in z between the center and surface is over twice as great for the realistic model than for the uniform density model.

It bears mentioning that, the curve for the Sun (or Earth) beyond the surface is independent of the internal density structure. This can be seen in the Schwarzschild metric previously reduced for constant t, θ, and ϕ; outside the object, $m(r) = M$ for all r, so

$$ds^2 = \frac{dr^2}{1 - 2M/r} \qquad * \tag{20}$$

has no dependence on $m(r)$ beyond dependence on the total mass M.

Also, if the above equation for z is written in conventional units the dependence on the object's radius and mass can be reduced to simple dependence on density:

$$z(r) = \sqrt{\frac{3c^2}{8\pi G\rho}}\left(1 - \sqrt{1 - \frac{8\pi G\rho}{3c^2}r^2}\right). \tag{21}$$

8 Comment on neutron stars

Previously we obtained an equation for s for an object of uniform density. A neutron star of radius $R = 10$ km and mass $M = 1.5M_{Sun}$ has $R_S = 4.5$ km. (In this case, R is not significantly greater than R_S so the approximation $\Delta R = R_S/3$ cannot be used.) Assuming uniform density, the equation gives s = 11 km, or ΔR=1 km= $0.1R$.

To estimate the relativistic volume of a neutron star, a numerical integration was performing using the above data and a step with $dr = 0.01R$. This gave a value for V_{rel} about 18% greater than the Euclidean volume. This suggests that relativistic space-time curvature is a significant consideration when modelling the interiors of neutron stars.

9 Conclusion

This exercise produced several expressions of relativistic curvature for solar system objects. The true diameters of the Sun and Earth are 4.1 km and 4.4 mm greater, respectively, than one would expect from applying Euclidean geometry ($C = \pi d$) to the observed surface of these bodies. These results are significantly affected by the non-uniform internal density variation of these bodies; they are 4 and 1.5 times greater, respectively, than for a equal mass/equal circumference object of uniform density. In the case of the Sun, this internal space-time curvature affords it a volume 6 parts per million greater than the Sun's surface would enclose in Euclidean space. An embedding diagram was graphed for the case of the Sun. This demonstrates the contrast between cmvatme in the Sun and in a uniform density model of the Sun. Quick calculations for a neutron star, assuming uniform density, showed the relativistic radius and volume to be 10% and 18% greater than the corresponding Euclidean values.

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`http://www.johnstonsarchive.net/relativity/stcurve.pdf`